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AN INTERNAL CHARACTERIZATION OF UNIFORM WEIGHT

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At different times, Mrowka and Juhasz have defined closely related cardinal invariants on topologies which tell how many pseudometrics are required to generate them. These invariants have external definitions—they are not defined in terms of the topology itself. The metrization number of Hodel is equal to these for normal topologies, though not in general. Below we give an internal definition of the Mrowka-Juhasz number.

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cardinal invariant	μ-additive topology
continuity space	pseudometric space
metrization number	uniform weight

1. Uniform weight and symmetric weight

We adopt the following notational conventions. Lower case greek letters early in the alphabet denote ordinals, those in the middle denote infinite cardinals; as usual, ω is the first infinite cardinal. Also $\inf \emptyset = \infty$.

We are interested in cardinal invariants related to metrization. For a topology T, Hodel's *metrization number* is $m(T) =$ least index of a discrete base for T (see [1]); here a *discrete base* is a collection $\boldsymbol{B} = \{B_\alpha : \alpha < \mu\}$ of subsets of T indexed by a cardinal μ, such that $\bigcup \boldsymbol{B}$ is a base for T and for each $x \in X$, $\alpha < \mu$, there is a neighborhood N of x disjoint from all but (at most) one element of B_α.

The *uniform weight* $u(T)$ of Juhasz [2] is the smallest possible cardinality of a base of a uniformity from which T arises. Essentially the same cardinal is the *Mrowka number* $M(T)$ [M]:= the smallest cardinal of a set of pseudometrics from which T arises (for metric spaces, $M(T) = 1$ and usually $u(T) = \omega$, but it is always true that $u(T) + \omega = M(T) + \omega$). Another definition of the same cardinal may be made in terms of continuity spaces since $u(T)$ is the least cardinal of a base for the set of positives in a symmetric continuity space from which T arises. (An elementary discussion of continuity spaces can be found in [5].)

Let B be a base of minimal cardinality for a uniformity U from which T arises. For each $u \in B$ there is a pseudometric d_u such that if $r > 0$ then $N_r \in U$, and $N_r \subset u$ (here $N_r = \{(x, y): d_u(x, y) \leq r\}$; see [3, pp. 184–188]). Thus $D = \{d_u: u \in B\}$ is a set of pseudometrics generating U, thus T, and $|D| \leq |B|$, therefore $M(T) \leq u(T)$.

Now let D be a set of pseudometrics of minimal cardinality giving rise to T, and let $U = \{u \subset X \times X$: for some $r > 0$ and finite $F \subset D$, $N_{r,F} \subset u\}$, where for $r > 0$, F a set of pseudometrics, $N_{r,F} = \{(x, y)$: if $d \in F$ then $d(x, y) \leq r\}$. A long but straightforward proof shows that U is a uniformity, T arises from U and $B = \{N_{1/n,F}: F \subset D$ finite, n a positive integer$\}$ is a base for U; clearly $|B| \leq |D|\omega$, so $u(T) \leq M(T)\omega$.

Mrowka [8] has shown that for normal topologies, $M(T)\omega = m(T)\omega$. This clearly fails for a regular topology T which is not completely regular, since T does not arise from any uniformity, thus $u(T) = \inf \emptyset = \infty$, but if B is any base for T, index B by its cardinality $B = \{P_\alpha: \alpha < \nu\}$; clearly $\boldsymbol{B} = \{\{P_\alpha\}: \alpha < \nu\}$, is a discrete base for T, so $m(T) \leq \nu < \infty$. Below we give an internal characterization of $u(T)$. The following lemma leads to it, by characterizing the regularity of a topology in terms of a property of a base for it. We leave its straightforward proof to the reader.

1. Lemma. *Let S be a base for topology T. Then T is regular iff whenever $x \in P \in S$ there is a $Q \in S$ and an $R \subset S$ such that $x \in Q \subset X - \bigcup R \subset P$.*

It is useful to have terminology for the above, thus we give the following definition.

2. Definition. If $x \in P \in T$ then (Q, R) is a *regularizer* for (x, P) if $x \in Q \subset X - \bigcup R \subset P$, and $Q \in T$, $R \subset T$.

$S \subset T$ is *self-regular* if whenever $x \in P \in S$ then there is a regularizer (Q, R) for (x, P) such that $Q \in S$, $R \subset S$.

The previous lemma and the fact that every topology has a base yields the following proposition.

3. Proposition. *Let T be a topology. The following are equivalent*:

(i) *T is regular,*

(ii) *T has a self-regular base,*

(iii) *every base of T is self-regular.*

We now apply the above to indexed bases.

4. Definition. Let μ denote an infinite cardinal. The ordinals α, β are *μ-near* if $\alpha + \mu = \beta + \mu$ (μ-nearness is easily seen to be an equivalence relation, and if δ is the smallest ordinal μ-near α, then $[\delta, \delta + \mu)$ is the μ-near equivalence class of α).

Let $\boldsymbol{B}=\{B_\delta\colon \delta<\nu\}$ be an indexed set of subsets of T. Then $B(\alpha,\mu)=\bigcup\{B_\beta\colon \beta$ μ-near to $\alpha\}$, $T(\alpha,\mu)$ the topology generated by $B(\alpha,\mu)$. Also $B(\alpha)=B(\alpha,\omega)$, $T(\alpha)=T(\alpha,\omega)$.

$\boldsymbol{B}$ is *μ-quickly self-discrete* (*self-locally finite*) if for each $\alpha<\nu$, $x\in X$, there's a $P\in T(\alpha,\mu)$ such that P intersects at most one element (a finite number of elements) of B_α.

$\boldsymbol{B}$ is *μ-quickly self-regular* if each $B(\alpha,\mu)$ is a self-regular base of $T(\alpha,\mu)$.

Quickly means ω-quickly.

The *symmetric degree* $s(T)$ is the inf of the set of indices of quickly self-discrete self-regular families $\boldsymbol{B}$ such that $\bigcup\boldsymbol{B}$ generates T.

5. Theorem. $s(T)\omega=u(T)\omega$.

Proof. Let $\boldsymbol{B}=\{B_\alpha:\alpha<\nu\}$ be a quickly self-discrete self-regular family such that $\bigcup\boldsymbol{B}$ generates T. For each $\alpha<\nu$, $\{B_\beta\colon \beta$ near $\alpha\}$ is a countable set, in fact a sigma-discrete base for the regular (since $B(\alpha)$ is self-regular) topology $T(\alpha)$. Thus by the Bing metrization theorem there is a pseudometric d_α giving rise to $T(\alpha)$. But then T arises from $\{d_\alpha\colon \alpha<\mu\}$, so $M(T)\leqslant s(T)$, $u(T)\leqslant M(T)\omega\leqslant s(T)\omega$.

Conversely, let $D=\{d_\alpha\colon \alpha<\nu\}$ be a set of pseudometrics from which T arises. By Bing, for each $\alpha<\nu$ there is a sigma-discrete base $\boldsymbol{C}=\{C_{\alpha,n}\colon n<\omega\}$ for the topology T_α generated by d_α. If $\nu>\omega$ then $f\colon \nu\times\omega\to\nu$ via $f(\alpha,n)=\omega\alpha+n$ is one-one onto, otherwise let $\nu=\omega$ (if necessary by giving elements of B many indices) and let $f\colon \nu\times\omega\to\nu$ be any map which is one-one onto. Now let $B_{f(\alpha,n)}=C_{\alpha,n}$. Note that for $\alpha<\nu$, m, $n<\omega$, $f(\alpha,m)$ is near $f(\alpha,n)$, so $\boldsymbol{B}=\{B_\beta\colon \beta<\nu\}$ is quickly self-discrete and (since T_α is regular), quickly self-regular. Thus $s(T)\leqslant M(T)\omega\leqslant u(T)\omega$. □

By Theorem 5, $s(T)<\infty$ iff T is completely regular. A proof similar to that of Theorem 5, using Nagata–Smirnov metrization results shows the following theorem.

6. Theorem. $u(T)\omega=\inf\{\nu$: *ν is the index of a quickly self-locally finite, quickly self-regular family which generates T*$\}\omega$.

We also have a characterization of the uniform weight in terms of sets of normal subtopologies whose join is T.

7. Theorem. $M(T)\omega=\inf\{\Sigma\{m(T_i)\colon i\in I\}\colon \{T_i\colon i\in I\}$ *a set of normal topologies whose join is* $T\}$.

Proof. ($\leqslant$) Suppose T is the join of the set of normal topologies $\{T_i\colon i\in I\}$. Then for each i, $m(T_i)=M(T_i)\omega$ (by [8], since each T_i is normal), so for each i let D_i be a set of pseudometrics generating T_i, $|D_i|\omega=m(T_i)$. Then $D=\bigcup\{D_i\colon i\in I\}$ yields T and $M(T)\omega\leqslant|D|\omega\leqslant\Sigma\{m(T_i)\colon i\in I\}$.

($\geqslant$) Let D be a set of pseudometrics giving rise to T, of minimal cardinality. Then for each $d \in D$, its pseudometric topology T_d is normal and $m(T_d) \leqslant \omega$ (since it has a sigma-discrete base, which is a countable quickly self-discrete, self-regular base). Also, T is the join of the T_d's, and $\Sigma\{m(T_d): d \in D\} \leqslant |D|\omega = M(T)\omega$. $\square$

2. Related questions

In this section we briefly discuss several related questions, some settled, some unsettled. First: are there analogues of our results for spaces with large additivity? Recall that a topology (uniformity) is *μ-additive* is it is closed with respect to intersection of fewer than μ of its elements.

8. Definition. A set S of subsets of X $(X \times X)$ *μ-generates* a [necessarily μ-additive] topology T (uniformity U) if $\{\bigcap Q: Q \subset S, |Q| < \mu\}$ is a base for $T(U)$.

The *μ-symmetric degree* $s_\mu(T)$ of a topology T is the inf of the set of indices of μ-quickly self-discrete self-regular families $\boldsymbol{B}$ such that $\bigcup \boldsymbol{B}$ μ-generates T.

The *μ-uniform weight* $u_\mu(T)$ of an μ-additive topology T is the inf of the set of cardinalities of sets which μ-generate uniformities from which T arises.

9. Theorem. $s_\mu(T)\mu = u_\mu(T)\mu$.

The proof, as well as the result, is analogous to what is found in Theorem 5. The place of the Bing metrization theorem in that proof is taken by its generalization to higher additivities, due to Nyikos and Reichel [9]. There are two apparent generalizations of Theorem 6 to the higher additivity case (depending on whether we use self-locally finite as it stands, or replace it with "self-μ-bounded", defined by replacing "a finite number of elements" with "fewer than μ elements" in the definition of "self-locally finite" in Definition 4); both generalizations hold (a generalization of the Nagata–Smirnov results to higher additivity situations may be found in Wang Shu-Tang [12]). Theorem 7 also generalizes with little difficulty.

We were originally hoping to get analogous results for quasimetrics and bitopological spaces, and here we find our first open question. Although several authors have worked on analogues of Nagata–Smirnov to bitopological spaces (Kelly [4], Lane [7], and Salbany [10] among others), an appropriate analogue is not yet in place, thus consideration of this case must be postponed. (In fact, a proper generalization of this work to the bitopological case would be equivalent to such an analogue.)

Finally, our symmetric degree is defined similarly to Hodel's metrization number, $m(T)$, and one must ask whether they are in general equal (modulo omega); i.e., is $s(T)\omega = m(T)\omega$, or equivalently by Theorem 5, is $u(T)\omega = m(T)\omega$? As pointed out earlier, Mrowka [8] has shown this for normal topologies, and it fails for regular topologies which are not completely regular. We would like to know whether it holds for all completely regular topologies. Each quickly self-discrete quickly self-regular base gives rise to a discrete base with the same index (if F is a finite subset of the index cardinal μ, let $C_F = \{\bigcap R: R$ is the range of a function f on F such that if $\alpha \in F$, $f(\alpha) \in B_\alpha\}$; now reindex $\{C_F: F \subset \mu$ finite$\}$ by μ); thus $m(T) \leqslant s(T)$. The Bing metrization theorem implies that if $m(T) = \omega$, then $M(T) = 1$ so $s(T) \leqslant \omega$; thus any counterexample must satisfy $\omega < m(T) < s(T)$ (in fact, any ν-additive example must satisfy $\nu < m(T) < s(T)$). Hodel [1] has shown that $w(T) = m(T) + d(T) = m(T) + c(T) = m(T) + L(T)$ (as usual, $w(T)$, $d(T)$, $c(T)$, and $L(T)$ denote the weight, density, cellularity, and Lindelof number of T; see [2]), and we have shown elsewhere [6] that $w(T) = s(T) + d(T) = s(T) + c(T) = s(T) + L(T)$; any of the latter equations implies that $s(T) \leqslant w(T)$, so if $m(T) < s(T)$ we must also have by the former equations, $w(T) = d(T) = c(T) = L(T)$. The referee has pointed out to us that any counterexample must also satisfy $\chi(T) < w(T)$ ($\chi(T)$ denotes the local neighborhood character, $= \sup\{\inf\{N_x$: N_x a neighborhood base about $x\}: x \in X\}$). This holds since $\chi(T) \leqslant m(T) \leqslant w(T)$ and $\chi(T) \leqslant s(T) \leqslant w(T)$. These necessary conditions (with one further argument shown below) exclude all examples of completely regular, non-normal topologies known to us.

If T is a singular cardinal version of the Dieudonne, plank [11, example 89, p. 108], then all of these necessary conditions are satisfied, but the following argument (kindly provided us by the referee) shows that $m(T) = s(T)$. If we use cardinals λ and μ with $\mathrm{cf}(\lambda) \leqslant \mathrm{cf}(\mu)$ it is routine to find $\mathrm{cf}(\lambda)$ discrete families of clopen sets which generate the topology, so both $m(T)$ and $s(T)$ (the latter because clopen partitions are self-regular) are $\leqslant \mathrm{cf}(\lambda)$. On the other hand it is easy to see that the clopen singletons cannot be written as the union of fewer than $\mathrm{cf}(\lambda)$ discrete collections, and so equality holds.

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